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TYCHONOV'S THEOREM FOR G-SPACES
(A NOTE ON A PAPER BY S.A. ANTONYAN)

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Tychonov's theorem for G-spaces (a note on a paper by S.A. Antonyan) *)

by

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ABSTRACT

In this note we present an English version of most of the results of a paper by S.A. Antonyan. We also generalize his results to the case of arbitrary locally compact sigma-compact groups. Briefly, it concerns a version for G-spaces of the well-known result that every Tychonov space can be embedded in a cube of the same weight.

KEY WORDS & PHRASES: *G-space, equivariant embedding, infinite dimensional separable Frechet space, compact convex set, linear action*

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*) This report will be submitted for publication elsewhere.

The aim of this note is to present a version of most of the result of the paper [1] in the English language. Briefly, it concerns a version for G-spaces of the well-known result that every Tychonov space of weight τ can be topologically embedded in I^τ, the product of τ copies of the unit interval I. We shall provide full proofs for our results (in [1], only a special case is proven without indication of proofs of the more general cases). Also, we generalize the results of [1] to arbitrary locally compact, sigma-compact groups (in [1], results are stated only for compact and for locally compact second countable groups). Finally, we point out some connections with related results.

The letter G shall always denote a topological group. For terminology and notation concerning G-spaces, we refer to [7].

THEOREM 1. *Let* G *be locally compact and sigma-compact. Then for every* G-*space* $\langle X,\pi\rangle$ *with* X *a compact Hausdorff space of weight* $w(X) =: \tau$ *there exists an action* $\tilde{\pi}$ *of* G *on* $\mathbb{R}^\tau$ *such that*

(i) *the cube* I^τ *is an invariant subset of* $\mathbb{R}^\tau$ *under this action;*

(ii) X *can equivariantly be embedded in* I^τ.

Moreover, there exists a linear structure on $\mathbb{R}^\tau$ *making* $\mathbb{R}^\tau$ *(with its ordinary product topology) a locally convex topological vector space such that*

(iii) I^τ *is a convex subset of* $\mathbb{R}^\tau$;

(iv) *the action* $\tilde{\pi}$ *is linear (i.e.* $\tilde{\pi}^t\colon \mathbb{R}^\tau \to \mathbb{R}^\tau$ *is linear for every* $t \in G$*).*

PROOF. We assume that X is not finite, so that $\tau \geq \aleph_0$ (if necessary, replace X by $X \cup I$ (disjoint union) and extend the action of G to this larger space such that all points of I remain invariant). The proof consists of several steps.

Step 1. Let $C_c(G)$ be the space of all real-valued continuous functions on G, endowed with the compact-open topology, and define an action ρ of G on $C_c(G)$ by $\rho^t f(s) := f(st)$ for $f \in C_c(G)$ and $s,t \in G$ (for ρ to be continuous it is essential that G be locally compact; cf. [7; 2.1.4]. Then $C_c(G)^\tau$ is also a G-space, the action of G on $C_c(G)^\tau$ being defined coordinate-wise by ρ. Since X can be embedded in $\mathbb{R}^\tau$, it follows from [7; 7.1.4] that, as a G-space, X has an equivariant embedding ϕ in the G-space $C_c(G)^\tau$.

Since G is locally compact, $C_c(G)$ and hence $C_c(G)^\tau$ are *complete* locally convex topological vector spaces. Since $\phi[X]$ is a compact subset of this space, also the closed convex hull K of $\phi[X]$ in $C_c(G)^\tau$ is compact [3; Chap. I, § 4, no. 1]. Moreover, the action of G on $C_c(G)^\tau$ is linear and continuous, and this implies that K is invariant in $C_c(G)^\tau$ under the action of G.

Resuming, we have a complete locally convex topological vector space $C_c(G)^\tau$, a linear action of G on it, and we have a compact convex invariant subset K in which X can equivariantly be embedded.

Step 2. This step consists in proving the following statement, which comprises essentially the main idea of [1]:

Let K_0 be an infinite-dimensional compact convex subset of a separable Frechet space E. Then there exists a homeomorphism $\psi: E \to \mathbb{R}^{\aleph_0}$ such that $\psi[K] = I^{\aleph_0}$.

The proof consists of a straightforward application of three results from infinite dimensional topology. First, by the Anderson-Kadec theorem, there exists a homeomorphism $\psi_1: E \to \mathbb{R}^{\aleph_0}$, and, second, by Keller's theorem [2; III. Thm. 3.1], there exists a homeomorphism $\psi_2: K_0 \to I^{\aleph_0}$. Now we have the homeomorphism $\psi_2 o \psi_1^{\leftarrow} | \psi_1[K_0]: \psi_1[K_0] \to I^{\aleph_0}$ between the compact subsets $\psi_1[K_0]$ and $I^{\aleph_0}$ of the infinite dimensional separable Frechet space $\mathbb{R}^{\aleph_0}$. According to a theorem of Klee [5], this homeomorphism has an extension to a homeomorphism $\eta: \mathbb{R}^{\aleph_0} \to \mathbb{R}^{\aleph_0}$. Now let $\psi := \eta o \psi_1$.

Step 3. Our topological group G is assumed to be sigma-compact, so $C_c(G)$ is a Frechet space. Now observe that we can write $\tau = \tau . \aleph_0$, so the index-set for the product $C_c(G)^\tau$ may assumed to be a disjoint union of τ copies of a given countable set. This fixes a homeomorphism

$$\Phi: C_c(G)^\tau \to \prod_{\lambda\in\Lambda} E_\lambda$$

where Λ is a set of cardinality τ and $E_\lambda = C_c(G)^{\aleph_0}$ for every $\lambda \in \Lambda$. From this description it also follows, that Φ is linear and that Φ is equivariant. For every $\lambda \in \Lambda$, let $\Phi_\lambda: C_c(G)^\tau \to E_\lambda$ be the composition of Φ with the canonical projection onto E_λ. If we put $K_\lambda := \Phi_\lambda[K]$, then K_λ is a compact convex invariant subset of E_λ.

Note that E_λ is an infinite-dimensional Frechet space (a product of countably many Frechet spaces) and we may assume that K_λ is also infinite dimensional. (If it is not, then proceed as follows: let $J \subseteq C_c(G)$ be the (invariant!) set of all constant functions on G with values in the interval I. Note, that J is homeomorphic with I so that, in particular, J is compact. Then $J^{\aleph_0}$ is a compact subset of $C_c(G)^{\aleph_0} = E_\lambda$, hence $K_\lambda \cup J^{\aleph_0}$ is compact. If we replace K_λ by the closed convex hull of $K_\lambda \cup J^{\aleph_0}$, then we obtain an infinite-dimensional compact convex subset of E_λ, which is still invariant under the action of G.)

It follows, that the closed linear subspace F_λ of E_λ generated by K_λ is an infinite dimensional Frechet space, invariant under the action of G. Moreover, since K_λ is separable (being compact and metrizable) F_λ is separable as well.

Resuming, we have for every $\lambda \in \Lambda$ an infinite-dimensional compact convex subset K_λ of a separable Frechet space F_λ. Moreover, G acts linearly on F_λ such that K_λ is an invariant subset of F_λ (the action of G on F_λ is, of course, the action which is inherited from the action of G on E_λ in which F_λ is an invariant subspace). Finally, note that the composition of ϕ (from Step 1 of the proof) and Φ is an equivariant embedding of X into the invariant compact convex subset $\prod_{\lambda\in\Lambda} K_\lambda$ of the linear G-space $\prod_{\lambda\in\Lambda} F_\lambda$.

Step 4. By Step 2, for every $\lambda\in\Lambda$ there exists a homeomorphism $\psi_\lambda : F_\lambda \to \mathbb{R}^{\aleph_0}$. with $\psi_\lambda[K_\lambda] = I^{\aleph_0}$. The maps ψ_λ define in the obvious way a homeomorphism $\Psi\colon \prod_{\lambda\in\Lambda} F_\lambda \to (\mathbb{R}^{\aleph_0})^\tau \cong \mathbb{R}^\tau$ such that $\Psi[\prod_{\lambda\in\Lambda} K_\lambda] = (I^{\aleph_0})^\tau \cong I^\tau$. If the linear structure and the action of G are carried over from $\prod_{\lambda\in\Lambda} F_\lambda$ to $\mathbb{R}^\tau$ via this homeomorphism Ψ, then it is clear that the properties (i) through (iv) of the theorem are valid. □

<u>REMARK 1</u>. In [1], the theorem is only proved for the case that G is compact and X is a compact metric space. In that case, one needs only step 1 and step 2 of the above proof (the case that X is finite is dealt with in a different way). Notice, that in [1] the embedding of X into a compact convex invariant set of a linear Frechet G-space (i.e. step 1 of the proof) is obtained in a different way, as follows: since G is compact (!), there exists an invariant metric d on X. Let $C_u(X)$ be the space of all continuous functions on X endowed with the topology of uniform convergence, and define an action σ of G on $C_u(X)$ by $\sigma^t f(x) = f(\pi^{t^{-1}} x)$ for $f \in C_u(X)$, $t \in G$, $x \in X$. Then $C_u(X)$ is a separable Frechet space (for separability, use the

Stone-Weierstrass theorem), σ is a linear action of G on $C_u(X)$ and, finally, X can equivariantly be embedded in $C_u(X)$ by means of the mapping $x \mapsto d(x,.)$: $X \to C_u(X)$ (that this mapping is equivariant follows from invariance of the metric d).

In [1], the above theorem (or rather, the stronger theorem 2 below) is stated without any proof for the case that G is locally compact and second countable.

2. For the case $\tau = \aleph_0$ the above theorem, as far as properties (i) and (ii) are concerned (so without the statements about the linear strucuture) follow easily from [9]. In that case, no assumptions about G need to be made. For a related result, see [2; VI. Cor. 7.1]. Compare also with [6; 3.6] (actually, Theorem 1 above is stronger than this result in [6] in that G is allowed to be only sigma-compact instead of second countable).

In theorem 1, the linear structure and the action of G in $\mathbb{R}^\tau$ depend on the given G-space $\langle X,\pi\rangle$. The following "universal" result generalises theorems 3,4 and 5 in [1] where only second countable locally compact groups or compact groups are considered.

THEOREM 2. *Let* G *be locally compact and sigma-compact. Then for every infinite cardinal number* $\tau \geq w(G)$, *the weight of* G, *there exists an action* $\tilde{\pi}$ *of* G *on* $\mathbb{R}^\tau$ *such that*

(i) the cube I^τ *is an invariant subset of* $\mathbb{R}^\tau$ *under this action;*

(ii) every G-*space* $\langle X,\pi\rangle$ *with* X *a Tychonov space of weight* $w(X) \leq \tau$ *can equivariantly be embedded in* I^τ.

Moreover, there exists a linear structure on $\mathbb{R}^\tau$ *such that* $\mathbb{R}^\tau$ *is a locally convex topological vector space and properties (iii) and (iv) of theorem 1 are valid.*

PROOF. Every G-space $\langle X,\pi\rangle$ with X a Tychonov space of weight $w(X) \leq \tau$ can equivariantly be embedded in $\langle C_c(G)^\tau,\rho\rangle$ (compare with step 1 of the proof of theorem 1; for the embedding, compactness of X need not be assumed). By [8], the G-space $\langle C_c(G)^\tau,\rho\rangle$ can equivariantly be embedded in a G-space $\langle X^*,\pi^*\rangle$, where X^* is a compact Hansdorff space of weight

$$w(X^*) \leq \max\{L(G), w(C_c(G)^\tau)\}.$$

Since $L(G) = \aleph_0$ and $w(C_c(G)^\tau) = \tau\, w(G) = \tau$, it follows that $w(X^*) = \tau$. Now apply theorem 1 to $\langle X^*, \pi^* \rangle$. $\square$

<u>REMARK</u>. Certain restriction on the group in theorem 2 seem inevitable. The following example arose in a discussion with Jan van Mill.

Let G be the full homeomorphism group of $\mathbb{Q}$, endowed with the discrete topology, and let G act on $\mathbb{Q}$ in the obvious way. Suppose that $\mathbb{Q}$ could be equivariantly embedded in a compact subset of $\mathbb{R}^\tau$ with $\tau = \aleph_0 = w(\mathbb{Q})$ and that the action of G on $\mathbb{Q}$ could be extended to an action of G on $\mathbb{R}^{\aleph_0}$. Then the closure X of $\mathbb{Q}$ in $\mathbb{R}^{\aleph_0}$ would be a compactification of $\mathbb{Q}$ such that every homeomorphism of $\mathbb{Q}$ extends to a homeomorphism of X. By [4], this would imply that $X \cong \beta\mathbb{Q}$, a contradiction ($\beta\mathbb{Q}$ cannot be homeomorphic with a subset of the metrizable space $\mathbb{R}^{\aleph_0}$).

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